Abstract

Polytopes are generalizations of polygons and polyhedra, which are polytopes of dimension two and three, respectively. As geometric objects with combinatorially interesting properties, polytopes have applications in diverse areas such as linear programming, optimization, physics, and topology. Recently, there is a heightened interest in studying polytopes associated to graphs. We will use the Laplacian matrix of a graph to form a polytope by considering the rows of the matrix as vertices of our polytope. This technique of generating polytopes was introduced in 2017, and there is much to be explored. In this project, we work to discover which families of graphs yield certain polytopes with desirable properties and why those properties are important. We also experiment with known operations on graphs and study their effects on the corresponding polytopes. Our methods include matrix manipulations, interpretation of lattice points as code words from coding theory, and computer computations.
1 Project Description

1.1 Introduction and Background

The combinatorial objects at the center of this project are polytopes and graphs, both of which are accessible to undergraduate students. Each graph is represented by a unique matrix which encodes its vertex-edge incidences, known as the Laplacian matrix of the graph. To obtain a polytope, we realize the rows of this matrix as vertices and consider the interior region. If the underlying graph is simple and connected with \( n \) vertices, the resulting polytope is a simplex, which is an \((n-1)\)-dimensional polytope with exactly \( n \) affinely independent vertices. If the underlying graph is directed, the resulting polytope may or may not be a simplex. These Laplacian polytopes live in \( \mathbb{R}^d \) for \( 0 < d < n \) and can be understood through their defining matrices.

Laplacian simplices were introduced by Braun and the applicant in [4] as a way to associate a polytope to a simple connected graph \( G \) through its Laplacian matrix. This construction is analogous to that of the edge polytope, the convex hull of the columns of the unsigned vertex-edge incidence matrix of a graph, which has been studied in detail over the past several decades; see [5, 7, 9, 10]. An important part of mathematics research is drawing connections between different fields. Laplacian simplices connect graph theory ideas with algebraic combinatorics.

Properties of the Laplacian simplex associated to \( G \), denoted \( \mathcal{P}_G \), are directly related to properties of \( G \). For instance, the normalized volume of \( \mathcal{P}_G \) is equal to the product of the number of vertices and the number of spanning trees of \( G \). It is known that certain families of graphs exhibit certain behaviors of \( \mathcal{P}_G \). For example, if \( G \) is a tree or complete graph, then \( \mathcal{P}_G \) is reflexive (defined in the following subsection). In the case of a cycle, \( \mathcal{P}_G \) is reflexive if and only if the cycle has an odd number of vertices. The interplay between the graphical structure of \( G \) and the geometric structure of \( \mathcal{P}_G \) has already produced interesting results related to reflexivity and other properties of polytopes. A natural generalization of \( \mathcal{P}_G \) is examined in [1] by considering the Laplacian simplex associated to a digraph \( D \). There is much to learn about how operations with digraphs affect their corresponding Laplacian simplices.

The search for reflexive simplices is a popular topic in the field of algebraic combinatorics. The number of reflexive simplices is known for small dimensions; however, past dimension four, the number of reflexive simplices is unknown. In addition to classifying reflexive simplices, it is of much interest to discover how this property relates to other attributes of the polytope. A well-known, long-standing conjecture in the field relates the property of reflexive to unimodular Ehrhart \( h^* \) vector [8]. This project contributes a broader context to study this open problem and may give insight on whether the conjecture is true.
Since this project is broadly defined, there are a few paths we can explore depending on the background of the student. The motivating theme is to explore the connections between polytopes and their underlying graphs. Specifically, we will look at the following research questions:

1. Which families of graphs yield reflexive simplices?
2. Can we characterize all reflexive Laplacian simplices based on a graphical property?
3. Which polytopes in the equivalence class of a Laplacian simplex are also Laplacian simplices? What are their underlying graphs?
4. Apply this construction using Cayley graphs (an unexplored graph type). What are the resulting simplices?
5. How many unimodularly equivalent Laplacian simplices are there of each dimension?
6. Consider the questions above in the case that the underlying graph is directed.

In this project, our goal is to completely answer or provide significant progress towards a solution to a few of the research questions. The student will write a paper carefully proving any new results. I plan to include any significant result in an upcoming paper to be published.

1.2 Proposed Research Plan

The methodology is grounded in Ehrhart theory, which was developed to study discrete properties of polytopes. A basic introduction is found [3]. Here are the basics needed to understand the two main features of this project: reflexive, and the $h^*$-vector. The convex hull of $d + 1$ integer points in $\mathbb{R}^d$ is called a lattice simplex if it has maximal dimension. Let $\{v_1, \ldots, v_{d+1}\}$ be the vertices of a simplex $\mathcal{P}$. The dual of $\mathcal{P}$ that contains $0$ in its interior is $\mathcal{P}^\vee := \{x \in \mathbb{R}^d \mid xy^T \leq 1 \text{ for all } y \in \mathcal{P}\}$. The fundamental parallelepiped of $\mathcal{P}$ is

$$
\Pi(\mathcal{P}) := \left\{ \sum_{i=1}^{d+1} \lambda_i (v_i, 1) \mid 0 \leq \lambda_i < 1 \right\},
$$

where $(v_i, 1)$ denotes the integer point $v_i$ with 1 appended at the end. The vector $h^*(\mathcal{P}) = (h_0, \ldots, h_d)$, where

$$
h_i(\mathcal{P}) = |\Pi(\mathcal{P}) \cap \{x \in \mathbb{Z}^{d+1} \mid x_{d+1} = i\}|,
$$

is called the $h^*$-vector of $\mathcal{P}$. The simplex is called reflexive if $h^*(\mathcal{P})$ is symmetric, that is, $h_i = h_{d-i}$. Geometrically, a polytope containing $0$ is reflexive if its dual is a lattice polytope. We use a computer software to experimentally compute the $h^*$-vectors of Laplacian simplices, but then must prove such polytopes are reflexive. To do this, we rely on the interpretation of lattice points in $\Pi(\mathcal{P})$ as code words.
Following the approach of Batyrev and Hofscheier [2], we consider the set

\[ \Lambda(\mathcal{P}) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_{d+1}) \mid \sum_{i=1}^{d+1} \lambda_i(v_i, 1) \in \Pi(\mathcal{P}) \cap \mathbb{Z}^{d+1} \right\}. \]  

(2)

Since \( \mathcal{P} \) is a lattice simplex, for \( \lambda \in \Lambda(\mathcal{P}) \) we have \( \lambda_i \in \mathbb{Q} \cap [0, 1) \). In fact, \( \Lambda(\mathcal{P}) \) is an abelian subgroup of \( \mathbb{Q}/\mathbb{Z} \), with addition given by

\[ (\lambda_1, \ldots, \lambda_{d+1}) + (\lambda'_1, \ldots, \lambda'_{d+1}) = (\{ \lambda_1 + \lambda'_1 \}, \ldots, \{ \lambda_{d+1} + \lambda'_{d+1} \}), \]  

(3)

where \( \{ \cdot \} \) denotes the fractional part of a number. Using this interpretation, lattice points in \( \Pi(\mathcal{P}) \) are more easily found by considering the generators of the abelian group. Classifications of families of Laplacian simplices can be stated in terms of \( \Lambda(\mathcal{P}) \), some of which are found [6].

Last summer I worked with a student who was a double major in math and computer science. He made progress working on research questions 1 and 2. We discovered a new family of reflexive simplices coming from windmill graphs, \( \mathcal{W}_{n,m} \). The student was also able to come up with and prove new graph operations that preserved the reflexive property of the Laplacian simplex. The student produced a computer program to determine the number of equivalence classes of Laplacian simplices for all Laplacian simplices of dimension up to \( n = 7 \). The input was a graph and the output was the equivalence class of the corresponding Laplacian simplex. The student also created a program to determine whether a Laplacian simplex is reflexive. Now we have a quick check to develop more examples from which to make conjectures about more families of graphs which produce reflexive simplices. This upcoming summer I hope to build on these results.

### 1.3 Mentorship Plan

I plan to work on this project along with the student throughout the summer. The student will be required to write a paper detailing their results along with mathematically sound proofs. We will meet virtually two times per week for the student to share their progress, ask any questions, and get my feedback on their writeup. Between meetings, I expect the student to spend sufficient time on our project.

At the beginning of our project, I will have the student read a chapter in [3], and we will meet to understand the basics of reflexive polytopes as well as the above methods. I will present this problem along with some known results, and I expect them to become familiar with the proposed questions so they may develop their own conjectures to answer these guiding questions. The first research task for the student is to use the code of the previous summer’s work to generate examples of simplices which are unimodularly equivalent. From there the student can make conjectures as to the number of equivalence classes of Laplacian polytopes of a fixed dimension. The student may choose to write an additional computer program,
which will further generate examples of Laplacian simplices of a given property.

A signature aspect of mathematics research is proof writing. The student is expected to construct mathematical proofs of their results. I will work with the student on the proof techniques needed and offer further direction when necessary. This project is broad enough to allow the student to explore their own curiosity. The student is expected to engage in this project outside of our designated meeting times.

Mathematical research requires creativity. As a mentor, I will encourage curiosity in my student. This project is designed to direct students to ask their own questions about these objects. In doing this, I hope to give students an experience of a pure mathematics research project. They will learn the process of generating small examples by hand or through use of a computer and proceeding to make conjectures about what is true in general. At the end of the summer, I expect the student to have new results of their own about graph operations and which families of graphs yield reflexive Laplacian simplices.

1.4 Proposed Research Plan Timeline with Project Goals

The student researcher will meet two times per week with me in addition to attending the SURE Wednesday seminars. The student will also meet with other math research students on a weekly basis to give a presentation on their progress. Below is a general time line.

Week 1: Assigned reading. Meet to discuss lattice polytopes, graphs, the basics of Ehrhart theory, additive groups, and the connection between fundamental parallelepiped points and code words. Student will work on routine problems in the area to get a taste of the mathematics used.

Week 2: Introduction of project. In this week, I will assign some reading for the student that lists a few known results.

Weeks 3-5: I will provide an introduction to Normaliz, which is open source software we use to compute the $h^*$-vector of a polytope. This will enable us to immediately determine whether a polytope is reflexive. This week the student will generate many examples and use these examples to make conjectures. We will try to prove conjectures. Students will keep an organized document of work, which I will check on a weekly basis.

Weeks 6-8: At this point, I expect student will have a few results of their own, typed up (including proofs). They will either continue looking at proposed research questions, or they will be equipped at this point to formulate their own research questions concerning these objects. There is an option to transition to directed graphs.

Week 9: Find an ending point and summarize results. Student may attend and present at
the 2021 MAA MathFest Conference, which is virtual this year.

**Week 10:** Finish the formal write up of results and prepare for SURE presentation. Possibly submit to a journal.

### References


## 2 Remaining Application Components

### 2.1 Description of Funding for Materials and Supplies

There is no additional funding for materials or supplies needed for this project.
2.2 Criteria for Student Applicants

This research involves linear algebra, graph theory, and potentially a small amount of computer programming. It is appropriate for students who have taken Linear Algebra or Applied Linear Algebra as well as Discrete Mathematics. Foundations of Advanced Mathematics is helpful, but not required.